Delayed feedback control of dynamical systems at a subcritical Hopf bifurcation

K. Pyragas,[1,2,*] V. Pyragas,[1] and H. Benner[2]
[1]*Semiconductor Physics Institute, LT-2600 Vilnius, Lithuania*
[2]*Institut für Festkörperphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany*
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We consider the delayed feedback control of a torsion-free unstable periodic orbit originated in a dynamical system at a subcritical Hopf bifurcation. Close to the bifurcation point the problem is treated analytically using the method of averaging. We discuss the necessity of employing an unstable degree of freedom in the feedback loop as well as a nonlinear coupling between the controlled system and controller. To demonstrate our analytical approach the specific example of a nonlinear electronic circuit is taken as a model of a subcritical Hopf bifurcation.

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Delayed feedback control (DFC) [1] is a convenient tool to stabilize unstable periodic orbits (UPO's) embedded in the strange attractors of chaotic systems. The method is reference free; it makes use of a control signal obtained from the difference between the current state of the system and the state of the system delayed by one period of the unstable periodic orbit. The method allows us to treat the controlled system as a black box; no exact knowledge of either the form of the periodic orbit or the system equations is needed. By giving only the period of the unstable orbit the system under control automatically settles on the desired periodic motion, and stability of this motion is maintained with only tiny perturbations. The delayed feedback control algorithm is especially superior for fast dynamical systems, since it does not require any real-time computer processing. Successful implementation of this algorithm has been attained in quite diverse experimental systems including electronic chaotic oscillators [2], mechanical pendulums [3], lasers [4], gas discharge systems [5], a current-driven ion acoustic instability [6], a chaotic Taylor-Couette flow [7], chemical systems [8], high-power ferromagnetic resonance [9], helicopter rotor blades [10], and a cardiac system [11].

Despite a certain progress in experiment the theory of delayed feedback control is far from being completed. This theory is rather intricate since it involves nonlinear delay-differential equations. Even linear stability analysis of the delayed feedback systems is difficult. The Floquet exponents (FE's) of periodic orbits controlled by the delayed feedback method are usually computed numerically [1,12,13]. Some analytical estimations [14,15] have been obtained for the case of unstable periodic orbits originating from a period doubling bifurcation. There are a few more examples where analytical asymptotic methods have been applied to time-delay systems. The synergetic approach [16] and the multiple-scaling method [17] were used to derive the normal form of the delay-induced Hopf bifurcation in the first-order phase-locked loop system with time delay. The multiple-scaling method has also been applied close to the first period doubling bifurcation in a model describing a delayed feedback control of a class-B laser [18].

A topological limitation of the delayed feedback control method has recently obtained much attention. It has been proven [14,19] that the method fails in the case of torsion-free periodic orbits or, more precisely, for unstable periodic orbits with an odd number of real positive Floquet exponents. A similar limitation emerges in the simpler problem of adaptive stabilization of unknown steady states of dynamical systems [20]. To overcome this limitation one of us (K.P.) has recently proposed to introduce an unstable degree of freedom into the feedback loop [21]. It was shown that such an unstable delayed feedback controller (UDFC) can stabilize an unstable periodic orbit of the Lorenz system. Unfortunately an analytical treatment of this system is hardly possible, and only numerical evidence has been presented in Ref. [21].

In this paper we consider the problem of stabilizing an unstable periodic orbit that appears in a dynamical system close to a subcritical Hopf bifurcation. This is the simplest situation giving rise to the topological limitation of the usual delayed feedback algorithm; an unstable periodic orbit emerging from this bifurcation is torsion free and, therefore, requires the use of an unstable controller. However, close to the bifurcation point the periodic orbit is only weakly unstable, and its stabilization is a relatively simple problem. The most important advantage of this situation is that the problem can be treated analytically by means of standard asymptotic methods developed in the theory of weakly nonlinear oscillators.

Nonlinear circuit as a model of a subcritical Hopf bifurcation. The problem of controlling an unstable periodic orbit at a subcritical Hopf bifurcation can be considered in a general way; however, for the clarity of presentation we restrict ourselves to a specific example of dynamical system shown in Fig. 1. The system represents a nonlinear circuit described by

$$L\dot{I} = -IR - V - f(I), \; C\dot{V} = I. \tag{1}$$

Here I is the current and V is the voltage on the capacitor C. The function $f(I)$ describes the voltage versus current char-

*Electronic address: pyragas@pfi.lt
URL: http://pyragas.pfi.lt

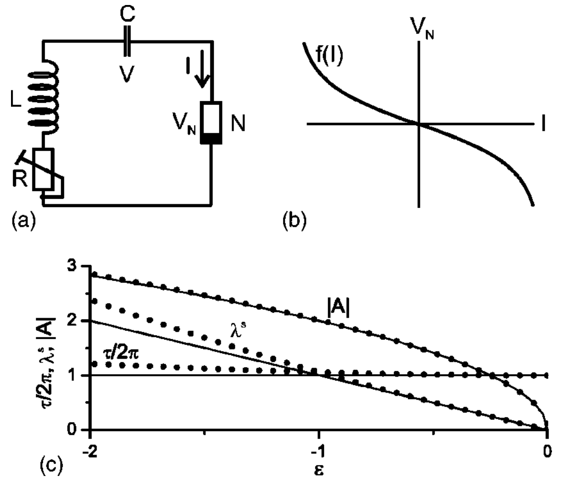

FIG. 1. (a) Circuit modeling a subcritical Hopf bifurcation. (b) Current vs voltage characteristic of the nonlinear element. (c) Amplitude $|A|$, period τ, and Floquet exponent λ^s of the unstable limit cycle as functions of the bifurcation parameter ε. Lines represent analytical results obtained from the averaged equation (5). Dots are the numerical results obtained from the exact equations (2). The amplitude is defined as the maximum of the x variable on the limit cycle.

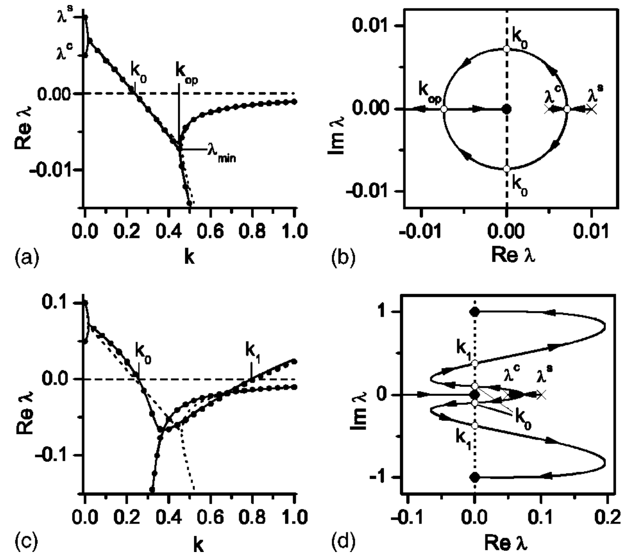

FIG. 2. (a) Real parts of leading Floquet exponents of the controlled UPO as functions of the control gain for $\varepsilon=-0.01$, $\lambda^c=0.005$. Dotted and solid lines show the solutions of the characteristic equations (10) and (11), respectively. Dots correspond to the values of Floquet exponents obtained from the exact variational equations (12). (b) Root loci of Eq. (11) as k varies from 0 to ∞ for the same parameter value as in (a). Crosses and black dots denote the location of the roots for $k=0$ and $k=\infty$, respectively. (c) and (d) Same diagrams as in (a) and (b) but for $\varepsilon=-0.1$ and $\lambda^c=0.05$.

acteristic $V_N=f(I)$ of a nonlinear element N placed in a series with the LC circuit. We assume that this element has a negative differential resistivity and for small I can be approximated by the function $f(I)=-aI-bI^3+O(I^5)$ with positive parameters a and b. Using the dimensionless variables $x=I/I_0$ and $y=V/V_0$, where $I_0=\sqrt{\rho/3b}$, $V_0=I_0\rho$, and $\rho=\sqrt{L/C}$, and normalizing the time to the characteristic period $T=\sqrt{LC}$ of the LC circuit, Eqs. (1) are simplified to

$$\dot{x}=-y+\varepsilon x+x^3/3,\ \dot{y}=x. \tag{2}$$

The only dimensionless parameter $\varepsilon=(a-R)/\rho$ can be easily controlled by varying the resistor R. The system (2) can be presented in the form $\ddot{x}+x-(\varepsilon+x^2)\dot{x}=0$ similar to the well-known van der Pol equation, with the only difference that the term $x^2\dot{x}$ comes here with a negative sign. For small ε, there are many mathematically rigorous ways (e.g., method of averaging, multiscale expansion, and other asymptotic methods) to obtain an approximate solution of this equation.

Defining the complex amplitude $A(t)$ by

$$y=(Ae^{it}+A*e^{-it})/2,\ x=(iAe^{it}-iA*e^{-it})/2, \tag{3}$$

and inserting them in Eqs. (2) we get

$$\dot{A}=\frac{A}{8}(4\varepsilon+|A|^2)-\frac{A^*}{8}(4\varepsilon+|A|^2)e^{-i2t}-\frac{A}{24}e^{i2t}+\frac{A^*}{24}e^{-i4t}. \tag{4}$$

Close to the bifurcation point $\varepsilon=0$, slow variations of the amplitude $A(t)$ can be determined by averaging Eq. (4) over the period of the fast oscillations, $\tau=2\pi$. This averaging is equivalent to neglecting the terms containing fast oscillations ($e^{\pm it}$, $e^{\pm i2t}$, etc.). Thus the averaged equation for the amplitude reads

$$\dot{A}=A(4\varepsilon+|A|^2)/8. \tag{5}$$

For $\varepsilon<0$, this equation has two steady-state solutions $A=0$ and $|A|=2\sqrt{-\varepsilon}$. The first represents a stable fixed point of the system at the origin $(x,y)=(0,0)$, and the second corresponds to an unstable limit cycle with the period $\tau=2\pi$, amplitude $2\sqrt{-\varepsilon}$, and a real positive Floquet exponent $\lambda^s=-\varepsilon$. For $\varepsilon>0$ the limit cycle disappears, and the fixed point at the origin becomes unstable. Thus at $\varepsilon=0$ we have a subcritical Hopf bifurcation. As is seen from Fig. 2(c), the analytical results obtained from the averaged equation (5) are in good quantitative agreement with the numerical results determined from the exact equations (2) when the system is in the vicinity of the bifurcation point.

Nonlinear delayed feedback controller. Now we assume that the current x is an observable accessible in experiment. To stabilize the unstable limit cycle appearing for $\varepsilon<0$ we consider the following delayed feedback control algorithm:

$$\dot{x}=-y+\varepsilon x+x^3/3+wx, \tag{6a}$$

$$\dot{y}=x, \tag{6b}$$

$$\dot{w}=\lambda^c w-k(x-x_\tau)x. \tag{6c}$$

The term wx in Eq. (6a) is the control perturbation introduced in the circuit as an additional voltage source. Equation (6c) describes an unstable delayed feedback controller with $\lambda^c>0$. Here w is a dynamical variable of the controller and k defines the feedback strength. We use the notation $x_\tau\equiv x(t-\tau)$. Note that the perturbation does not change the solution of the free system corresponding to the UPO of period τ, since, for $x=x_\tau$, Eq. (6c) is satisfied by $w=0$ and the perturbation wx in Eq. (6a) vanishes.

Unlike the control algorithm considered in Ref. [21] here we introduce nonlinear coupling terms—namely, the products wx and $(x-x_\tau)x$ in Eqs. (6a) and (6c), respectively. The nonlinearity is a necessary ingredient of the DFC algorithm when considering the stabilization of UPO's close to the bifurcation point. It is easy to verify that any linear coupling terms [e.g., w in Eq. (6a) and $x-x_\tau$ in Eq. (6c)] vanish due to the averaging procedure and thus result in uncoupled averaged equations for the slow dynamics of the controller and the controlled system. To provide an interrelation between these two subsystems in the averaged equations we need a nonlinear coupling in the original equations.

For small values of the parameters ε and λ^c, the averaged equations for the closed-loop system are obtained by inserting Eqs. (3) in system (6) and neglecting the fast-oscillating terms:

$$\dot{A} = A(4\varepsilon + |A|^2)/8 + Aw/2, \tag{7a}$$

$$\dot{w} = \lambda^c w - k(2|A|^2 - AA_\tau^* - A_\tau A^*)/4. \tag{7b}$$

Using the ansatz $A(t)=r(t)e^{i\varphi(t)}$, from the imaginary part of Eq. (7a) it is easy to derive an equation for the phase, $r\dot{\varphi}=0$. It follows that the phase is independent of time, $\varphi=\text{const}$. For the slowly varying real amplitude $r(t)$ and controller variable $w(t)$ we obtain

$$\dot{r} = r(4\varepsilon + r^2)/8 + rw/2, \tag{8a}$$

$$\dot{w} = \lambda^c w - kr(r - r_\tau)/2. \tag{8b}$$

This system can be even more simplified. Taking into account that $r(t)$ is a slow variable the delay term r_τ can be approximated by the first derivative, $r_\tau=r(t-\tau)\approx r(t)-\tau\dot{r}$. This approximation is valid for $\tau|\dot{r}|/r \ll 1$. Then the time-delay system (8) transforms to a system of ordinary differential equations:

$$\dot{r} = r(4\varepsilon + r^2)/8 + rw/2, \tag{9a}$$

$$\dot{w} = \lambda^c w - k\tau r\dot{r}/2. \tag{9b}$$

The eigenvalues λ of the fixed point $(r_0,w_0)=(2\sqrt{-\varepsilon},0)$ of this system satisfy the characteristic equation

$$\lambda^2 - (\lambda^c - \varepsilon + \varepsilon k\tau)\lambda - \varepsilon\lambda^c = 0. \tag{10}$$

They correspond to two leading nonzero FE's of the controlled UPO (the zero FE is defined by the equation for the phase $\dot{\varphi}=0$ derived above). Note that the UPO satisfying the time-delay system (6) has an infinite number of FE's, and most of them are lost in this approximation. A more precise characteristic equation for the FE's can be derived from the averaged equations (8) without using the approximation for the time-delay term r_τ. Linearization of Eqs. (8) around the fixed point $(r_0,w_0)=(2\sqrt{-\varepsilon},0)$ leads to the transcendental equation

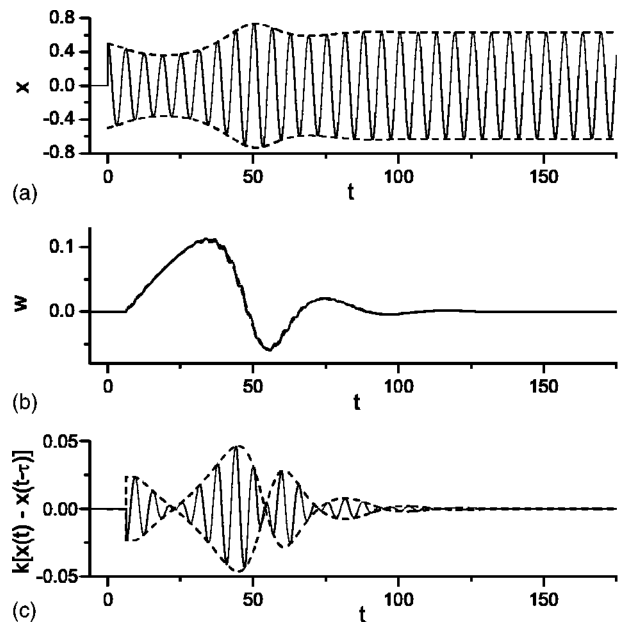

FIG. 3. Dynamics of (a) current x, (b) controller variable w, and (c) delayed feedback perturbation $k(x-x_\tau)$. Solid lines are the solutions of the nonlinear system (6) with initial conditions $x(t)=0$ for $-\tau\leq t<0$, $x(0)=0.5$, $y(0)=0$, and $w(0)=0$. Dashed lines represent the solution of averaged equations (8) with initial conditions $r(t)=0$ for $-\tau\leq t<0$, $r(0)=0.5$, and $w(0)=0$. The values of parameters are $\varepsilon=-0.1$, $\lambda^c=0.05$, $\tau=6.2871$, and $k=0$ for $t<\tau$ and $k=0.35$ for $t>\tau$.

$$\lambda^2 - (\lambda^c - \varepsilon)\lambda - \varepsilon\lambda^c - \varepsilon k(1 - e^{-\lambda\tau}) = 0. \tag{11}$$

For $|\lambda|\tau\ll 1$, it coincides with Eq. (10) due to the approximation $e^{-\lambda\tau}\approx 1-\lambda\tau$. In Figs. 2(a) and 2(c) we compare the FE's defined by Eqs. (10) and (11) with the "exact" values of the FE's obtained numerically from the nonaveraged variational equations

$$\delta\dot{x} = -\delta y + [\varepsilon + x_0^2(t)]\delta x + x_0(t)\delta w, \tag{12a}$$

$$\delta\dot{y} = \delta x, \tag{12b}$$

$$\delta\dot{w} = \lambda^c\delta w - kx_0(t)(\delta x - \delta x_\tau), \tag{12c}$$

derived from the original system (6). Here $\delta x=x-x_0(t)$ and $\delta y=y-y_0(t)$ are small deviations from the periodic orbit $[x_0(t),y_0(t)]=[x_0(t+\tau),y_0(t+\tau)]$, which satisfies the free system (2), and $\delta w=w$.

For $|\varepsilon|\tau\ll 1$, all three above results are in good quantitative agreement [Fig. 2(a)]. Thus the leading FE's of the controlled UPO can be reliably obtained from the simple quadratic equation (10). The stability conditions of this equation for $\varepsilon<0$ are

$$\lambda^c > 0,\ k > k_0 = (\lambda^c - \varepsilon)/(-\varepsilon\tau). \tag{13}$$

The first condition confirms the general statement that the torsion-free UPO's can be stabilized only with an unstable controller. The second condition can be rewritten in the form $k\tau>1+\lambda^c/\lambda^s$, where λ^c is the eigenvalue of the free controller and $\lambda^s=-\varepsilon$ is the FE of the unstable limit cycle of the free system. The mechanism of stabilization is evident from Fig. 2(b). For $k=0$, two real positive solutions of Eq. (10), $\lambda=\lambda^s$ and $\lambda=\lambda^c$, describe unstable eigenvalues of the free system and the free controller, respectively. With increasing k, the eigenvalues approach each other on the real axis, and then collide and pass to the complex plane. For $k=k_0$, they cross the imaginary axis and move symmetrically into the left half-plane; i.e., both the system and the controller become stable. An optimal value of the control gain is $k_{op}=k_0+2\sqrt{\lambda^c/\lambda^s}/\tau$ since it provides the fastest convergence to the stabilized UPO with the characteristic rate $\lambda_{min}=-\sqrt{\lambda^s\lambda^c}$.

For large values of $|\varepsilon|$, the root locus diagram is more complicated; see Fig. 2(d). For $|\varepsilon|\tau\sim 1$ the approximation of the delay term r_τ by the derivative is not valid; however, for $|\varepsilon|\ll 1$ we can use the averaged Eq. (8) as well as the transcendental characteristic Eq. (11). Figure 2(c) shows that Eq. (11) indeed yields good quantitative results, while Eq. (10) is no longer valid. Now the eigenvalues due to the delay term come into play. As a result, there appears a second stability threshold k_1 such that the stabilization of the UPO becomes possible only in a finite interval of the control gain, $k_0<k<k_1$.

Direct integration of the nonlinear equations (6) confirms the results of linear analysis. Figure 3 shows the successful stabilization of the UPO close to the bifurcation point. After a transient process the controlled system approaches the previously unstable orbit and the feedback perturbation vanishes. The envelopes of the transient are well described by the averaged equations (8). This confirms the validity of the averaging procedure applied to the time-delay system (6).

In conclusion, we have developed an analytical approach for the time-delayed feedback control of an unstable periodic orbit without torsion, which could not be stabilized by the conventional delay technique. As an example we have considered the specific model of a nonlinear electronic circuit at a subcritical Hopf bifurcation. Nevertheless, the described theoretical approach is valid for any dynamical system close to the bifurcation point and allows a complete analytical treatment. We believe that these results are of general importance for optimizing the control technique and will stimulate the search for further modifications aiming at the improvement of the control performance.

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